MTH 166

Lecture-8

Linear Differential Equations (LDE)-II

Topic:

Linear Differential Equations (LDE)

Learning Outcomes:

1. Linear Dependence and Independence of functions (Solutions) via Wronskian.

2. Abel’s Formula to find Wronskian.

3. Principle of Superposition.

Linear Dependence and Independence of Functions or Solutions:

Let (y_1, y_2, y_3) be the given set of functions or solutions.

Wronskian, $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$

(I) If $\boldsymbol{W = 0}$, then functions (y_1, y_2, y_3) are said to be **Linearly Dependent**.

(II) If $\boldsymbol{W \neq 0}$, then functions (y_1, y_2, y_3) are said to be **Linearly Independent**.

Fundamental Solutions or Basis:

The solutions which are linearly independent are called as Fundamental solutions or Basis.

Problem 1: Show that the functions: $(1, \sin x, \cos x)$ are linearly independent.

Solution: Let $(y_1, y_2, y_3) = (1, \sin x, \cos x)$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix}$

Expanding by first column:

$W = 1(-cos^2x - sin^2x) - 0 + 0 = -1(cos^2x + sin^2x) = -1$

Since $W \neq 0$, so the given functions are linearly independent.

Problem 2: Show that the functions: (x, x^2, x^3) are linearly independent on an interval which does not contain zero.

Solution: Let $(y_1, y_2, y_3) = (x, x^2, x^3)$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1{}' & y_2{}' & y_3{}' \\ y_1{}'' & y_2{}'' & y_3{}'' \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

Expanding by first column:

$W = x(12x^2 - 6x^2) - 1(6x^3 - 2x^3) + 0 = 2x^3 \neq 0$ (because $x \neq 0$)

Since $W \neq 0,$ so the given functions are linearly independent.

Problem 3: Show that the functions: $(2x, 6x + 3, 3x + 2\ )$ are linearly dependent.

Solution: Let $(y_1, y_2, y_3) = (x, x^2, x^3)$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1{}' & y_2{}' & y_3{}' \\ y_1{}'' & y_2{}'' & y_3{}'' \end{vmatrix} = \begin{vmatrix} 2x & 6x+3 & 3x+2 \\ 2 & 6 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$

(If one row of a determinant is zero, then value of determinant is always zero.)

Since $W = 0,$ so the given functions are linearly dependent.

Abel's Formula to find Wronskian:

Let us consider a 2nd order homogeneous LDE:

$$a_0 y'' + a_1 y' + a_2 y = 0 \qquad (1)$$

Where $a_0 \neq 0$, a_1, a_2 are continuous on an interval I and y_1, y_2 be its linearly independent solutions, then Wronskian is given as:

Wronskian, $\boldsymbol{W = ce^{-\int\left(\frac{a_1}{a_0}\right)dx}}$ where c is a constant.

<u>Problem 1.</u> Using Abel's formula, find Wronskian for: $y'' - 4y' + 4y = 0$

<u>Solution:</u> The given equation is: $y'' - 4y' + 4y = 0$ (1)

Comparing it with: $a_0 y'' + a_1 y' + a_2 y = 0$

Here $a_0 = 1$, $a_1 = -4$, $a_2 = 4$

By Abel's formula, Wronskian is given by:

$W = ce^{-\int\left(\frac{a_1}{a_0}\right)dx}$ where c is a constant

$$\Rightarrow W = ce^{-\int\left(\frac{-4}{1}\right)dx}$$

$$\Rightarrow W = ce^{4\int dx}$$

$$\Rightarrow W = ce^{4x}$$ **Answer.**

Problem 2. Using Abel's formula, find Wronskian for: $y'' + a^2 y = 0, a \neq 0.$

Solution: The given equation is: $y'' - 4y' + 4y = 0$ (1)

Comparing it with: $a_0 y'' + a_1 y' + a_2 y = 0$

Here $a_0 = 1, \;\; a_1 = 0, \;\; a_2 = a^2$

By Abel's formula, Wronskian is given by:

$W = ce^{-\int\left(\frac{a_1}{a_0}\right)dx}$ where c is a constant

$\Rightarrow W = ce^{-\int\left(\frac{0}{1}\right)dx}$

$\Rightarrow W = ce^{0} = c(1)$

$\Rightarrow W = c$ **Answer.**

<u>Principle of Superposition:</u>

If functions $(y_1, y_2, ---, y_n)$ are the solutions of homogeneous LDE:

$$a_0 y^n + a_1 y^{n-1} + ----- + a_{n-1} y' + a_n y = 0 \quad (1)$$

Then, their linear combination: $(c_1 y_1 + c_2 y_2 + --- + c_n y_n)$ is also a solution of LDE (1).

<u>Note:</u> Principle of superposition is not applicable to non-homogeneous LDE.

Problem 1: Show that (e^x, e^{-x}) and their linear combination $(c_1 e^x + c_2 e^{-x})$ are the solutions of homogeneous equation: $y'' - y = 0$. Also show that (e^x, e^{-x}) form basis or Fundamental solution.

Solution: The given homogeneous LDE: $y'' - y = 0$ (1)

Let $(y_1, y_2) = (e^x, e^{-x})$ be the given set of functions.

Part 1: Now they will be solutions of equation (1) if they satisfy equation (1).

i.e. y_1 will be solution of equation (1) if ${y_1}'' - y_1 = 0$ (2)

Here $y_1 = e^x \quad \Rightarrow {y_1}' = e^x \quad \Rightarrow {y_1}'' = e^x$

Substitute these values of y_1 and ${y_1}''$ in equation (2), we get:

$e^x - e^x = 0$ which is true.

So, y_1 is a solution of equation (1)

Now y_2 will be solution of equation (1) if $y_2'' - y_2 = 0$ (3)

Here $y_2 = e^{-x}$ $\Rightarrow y_2' = -e^{-x}$ $\Rightarrow y_2'' = e^{x}$

Substitute these values of y_2 and y_2'' in equation (3), we get:

$e^{x} - e^{x} = 0$ which is true.

So, y_2 is a solution of equation (1)

Part 2: By principle of superposition, if y_1 and y_2 are solutions, then their linear combination $(c_1e^{x} + c_2e^{-x})$ will also be a solution.

Let us verify it:

Let $y_3 = (c_1e^{x} + c_2e^{-x})$ $\Rightarrow y_3' = (c_1e^{x} - c_2e^{-x}) \Rightarrow y_3'' = (c_1e^{x} + c_2e^{-x})$

Now y_3 will be solution of equation (1) if $y_3'' - y_3 = 0$

i.e. $(c_1e^{x} + c_2e^{-x}) - (c_1e^{x} + c_2e^{-x}) = 0$ which is true.

Part 3:

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$= -e^x . e^{-x} - e^x . e^{-x}$$

$$= -2e^x \, e^{-x} = -2e^{x-x}$$

$$= -2e^0 = -2(1)$$

$$= -2 \neq 0$$

Since W ≠ 0, So, Solutions (y_1, y_2) are linearly independent.

Hence (e^x, e^{-x}) form basis or fundamental solutions of equation (1).

Problem 2: Show that $(1, x^2)$ form a set of fundamental solutions (basis) of homogeneous equation: $x^2y'' - xy' = 0$.

Solution: The given homogeneous LDE: $x^2y'' - xy' = 0$ (1)

Let $(y_1, y_2) = (1, x^2)$ be the given set of functions.

Part 1: Now they will be solutions of equation (1) if they satisfy equation (1).

i.e. y_1 will be solution of equation (1) if $x^2{y_1}'' - x{y_1}' = 0$ (2)

Here $y_1 = 1$ $\Rightarrow {y_1}' = 0$ $\Rightarrow {y_1}'' = 0$

Substitute these values of y_1 and ${y_1}''$ in equation (2), we get:

$x^2(0) - x(0) = 0$ which is true.

So, y_1 is a solution of equation (1)

Now y_2 will be solution of equation (1) if $x^2 y_2'' - x y_2' = 0$ (3)

Here $y_2 = x^2$ $\Rightarrow y_2' = 2x$ $\Rightarrow y_2'' = 2$

Substitute these values of y_2 and y_2'' in equation (3), we get:

$x^2(2) - x(2x) = 0$ which is true.

So, y_2 is a solution of equation (1)

Part 2: Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} 2x \neq 0$

Since W $\neq 0$, So, Solutions (y_1, y_2) are linearly independent.

Hence $(1, x^2)$ form basis or fundamental solutions of equation (1).